from (13), with an additional equation arising from (13) for $j = 0$ when u_{-1} is obtained from (14). The remaining equation is that for $\dot{z}$, Eq. (9).

Most computing machines are equipped with powerful subprograms for solving systems of ordinary differential equations. These programs are usually designed to select automatically the interval of integration based on criteria designed to bound the truncation error. Since the problems under consideration are often characterized by rapidly decaying transients, it follows that these problems allow vastly differing integration intervals to maintain a fixed truncation error. Explicit finite-difference methods require adjusting the space increment when varying the time increment to maintain a fixed stability ratio. For this reason, and because of a lack of criteria for adjusting step size in both explicit and implicit difference schemes, the method of lines seems to be the simplest, most efficient, and most flexible method for the type of problem presented herein. The greatest advantage, however, is its ability to find the position of the moving boundary without iteration. Variations of the procedure described in this paper have been used successfully by the author on problems involving the recession of a burning fluid and the solidification of explosives. In addition, the method can easily be extended to heat flow problems involving composite materials or to materials having thermal properties dependent on temperature.

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Reply by Author to H. J. Breaux

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In his comment[1] on Ref. 2, Breaux stated that the method presented was restricted by several conditions. The method in itself is not restricted, as claimed by Breaux; only the presentation in the Note was restricted for purposes of brevity.

A footnote[2] in my Note stated that the method has been extended to encompass radial heat flow in cylinders and spheres, finite-difference methods for inward and outward surface recession, equations describing cylindrical and spherical sublimation-conduction for several typical composite material arrangements, equations defining criteria for stopping surface recession, and equations to determine heat flow after recession terminates. These techniques all have been used successfully in the past and no shortcomings nor difficulties have been encountered.

As stated in Ref. 2, the equations presented are applicable only to the first interior nodal point where this point is located less than one Δ space increment from the receding surface. The surface temperature is constant (variable with time if one desires) and the other nodal points, with the exception of the second nodal point, are calculated using general explicit finite-difference equations. Once the recession front reaches the original first interior nodal point (T_2), then the equations presented[2] are applied to the original third nodal point (T_3), as stated on p. 1680[2] after Eq. (3). This procedure of handling the first interior nodal point temperature is continued until the receding surface is within one Δ increment of the back surface or an interface as stated in the Note[2] on p. 1680. Special equations, of no greater complexity, must then be applied.

The reason that the presentation was restricted to cases where the temperature of the receding surface remained constant during the recession process and the material thermal properties did not vary with temperature was to allow for comparisons of data to exact solution results. Variable thermal property data have been used and the receding surface temperature varying with time can be very easily incorporated.

To circumvent compressing the grid and encountering small time increments, especially as the receding surface approaches a backside or substructure interface, the nonshifting grid technique was considered advantageous. Inherent to a nonshifting grid is computational efficiency and a fixed stability ratio. If the computed temperatures are to be compared with empirical thermocouple data from a receding material, the fixed grid is most desirable and simple to arrange.

As stated in the Note,[2] the procedure requires a negligible increase in computer time over an identical nonrecession case, primarily because all calculations internal to the first interior node are made by ordinary explicit finite differences. The computer program efficiency is thus comparable to a nonrecession explicit finite difference program. The efficiency has been verified by past experience in using the program.

This recession-conduction procedure is extremely simple for extending general explicit finite-difference heat conduction routines to provide analytical capability in calculating the temperature of a structure during surface recession.

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Comment on "Inner Region of Transpired Turbulent Boundary Layers"

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In a recent Note Stevenson[1] discusses the disparity between the theoretical friction factors for the transpired turbulent boundary layer predicted in Ref. 2 and those in Ref. 3. The

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theories of both references are based in part on generalizations of experimental "inner region" velocity profiles and wall shear determinations. It follows logically that the theoretical discrepancy is due in part to a disparity in the experimental data. Through an assertion of the accuracy to which $d\theta/dx$ can be determined experimentally (± 0.00015), Stevenson demonstrates the possibility that the consequent difference in the mathematical formulation of the behavior of the inner region lies outside the uncertainty in the experimental data. With this argument as a basis, he suggests variations between experimental surface roughnesses and porosities as the causes for the discrepancies.

Although it is undeniable that surface-roughness effects are important, with and without mass addition, Stevenson's argument hangs on a most tenuous thread. Experimental friction factors determined from the two-dimensional integral momentum equation depend not only on the accuracy of $d\theta/dx$, but also on the accuracy of blowing rate. The accuracy of the effective two-dimensional $d\theta/dx$ is dependent on a myriad of experimental considerations, not the least of which are three-dimensional and end effects. Unfortunately, these effects tend to be amplified by mass addition.[6,17] In turn, a further amplification of error occurs in the subsequent evaluation of friction factor $C_f/2$. For example, at a blowing rate v_w/u_1 of 0.005 and a flat plate length Reynolds number of 10^6, $\pm 1\%$ uncertainty in both $d\theta/dx$ and v_w/u_1 admits to about $\pm 32\%$ uncertainty in friction factor. To demonstrate these uncertainties more graphically, Fig. 1 indicates chronologically the reported values of friction factor for this condition, all data in the figure being obtained employing the same MIT apparatus and the same experimental surface.

With regard to the particular data upon which Stevenson's formulation for the inner region is based, his were obtained in a fully developed transpired region whose length is on the order of only 1 ft, and the friction factor data of McQuaid[8] vary by a factor of 2 to 4 at a momentum thickness Reynolds number of about 4.5×10^3, and $v_w/u_1 \approx 0.0033$ (see Ref. 9). According to Stevenson's criterion on the accuracy of $d\theta/dx$, McQuaid's friction factors should be certain and in agreement within about $\pm 10\%$.

The theory of Ref. 2 is based on evaluations of all significant mass, momentum, and energy boundary-layer data obtained on the MIT apparatus prior to 1963,[4–7,10–13] as well as the impervious wall data of Refs. 14–16, and the suction data of Dutton.[18] It was concluded that for the MIT apparatus, no valid estimates of friction factor could be obtained at blowing ratios greater than 0.003, using the momentum equation. Therefore, the data presented in Ref. 2 are based on detailed evaluations of velocity profiles near the wall as described in Appendix A of that reference. Recently, data obtained at Stanford[9] using a new and carefully diagnosed apparatus (determined to have an aerodynamically smooth surface) have been interpreted by the momentum equation (at $Re_x = 10^6$, $v_w/u_1 = 0.0038$, $C_f/2 \approx 7 \times 10^{-4}$; which is about 40% higher than the theoretical results of Ref. 2). Reference 9 finds the second hypothesis of Rubesin [Stevenson's Eq. (5), u_a/u_τ = constant] serves to generalize their inner-region data, in substantial agreement with the findings of Kendall,[7] and in substantial agreement with the results of Van Driest's hypothesis [Stevenson's Eq. (7)]. These results are in conflict with Stevenson's suggested Eq. (6) (B = constant).

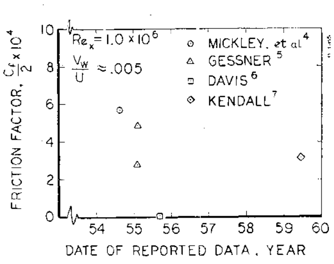

Fig. 1 Experimental friction factors obtained using the Massachusetts Institute of Technology apparatus.

The adequate mathematical formulation of the inner-region behavior requires consideration of the energy and mass boundary-layer behaviors as well. In this regard, the helium injection data of Kendall[7] and the heat-transfer data of Curl[10] were found to be quite predictable, employing the continuous mixing length model of Ref. 2, along with application of the turbulent Schmidt and Prandtl numbers of 0.75, which were interpreted from the experimental data. This success is particularly noteworthy in view of the low Schmidt number for helium (≈ 0.2).

Considering high transpiration rates, Stevenson introduces "dimensional reasoning" to establish that C is a constant in his Eq. (11). (His Fig. 2 demonstrates that C is obviously not constant.) It appears that this reasoning is predicated on the assumption that wall shear is identically and irrevocably zero, i.e., no longer a boundary-layer parameter. This appears unacceptable, at least to those of us who expect shear to approach zero asymptotically as transpiration increases in a favorable or negligible pressure gradient flow. Stevenson's subsequent discussion of the validity of the wall law expressions ceases to be pertinent if the dimensional reasoning cannot be accepted.

In conclusion, it is our opinion that 1) in the experiments referenced, the uncertainties in experimental friction factors obtained by using the two-dimensional momentum equation are significantly greater than stated or inferred by Stevenson; 2) the theory of Ref. 2 is felt to be more firmly based because consideration was given to data of many (rather than two) investigators, and consideration was given to mass and energy data in addition to momentum data; 3) recent data of Ref. 9 obtained on a well-qualified experimental system appear to support the inner-region formulations of Ref. 2, in conflict with the formulation of Ref. 3; and 4) the dimensional reasoning of Stevenson with regard to the inner-region behavior with high blowing is in error, and thus, his subsequent inferences are not pertinent.

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Reply by Author to T. J. Dahm and R. M. Kendall

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THE dimensional analysis in my Note was based on zero skin friction, a criterion used by Kutateladz and Leont'ev. In the discussion that followed I examined the implications of zero skin friction. The comment by Dahm and Kendall that "Fig. 2 demonstrates that C is obviously not constant" is incorrect. The figure does not prove the case one way or another; it was not intended to. If a breakaway had occurred, the value of the intercept on the horizontal would have continued to increase as it does on the plot of u/u_1 vs y/δ as it goes through separation in an adverse pressure gradient without injection. However, this type of separation is probably unlikely in the case of injection, and the skin friction probably approaches zero asymptotically.

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Admittedly the accuracy of v_w/u_1 is important in the evaluation of the skin friction, but the errors in v_w/u_1 are more readily estimated than the hidden errors in $d\theta/dx$. The criterion on the error of $d\theta/dx$ which was used to plot the chain dotted line in Fig. 1 of my Note spans the scatter of the data in the main experiments of McQuaid[1]; however, I agree that the criterion is too low. Many of the comments on the Note rely on being able to disregard the experimental results of McQuaid.[1] It is stated that there is a discrepancy in the results. In fact the "discrepancy" does not seem to be in the main set of experiments but in an experiment on a boundary layer with a discontinuity in the injection rate. McQuaid[1] states that this particular experiment "should be regarded as no more than preliminary in nature" because "the velocity distribution along the test wall could not be adjusted to a constant value to the same precision as in the main experiments" which "rendered the determination of c_f from the measured growth of θ very inaccurate." To disregard the main experiments because of difficulties in the discontinuous injection experiment seems unreasonable.

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Erratum: "Pressure Distribution on Cone-Cylinders in Hypersonic Flow"

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IN the above paper, the computations for Fig. 2 included the effects of the boundary layer. For the calculations of Ref. 5 of that paper, the boundary layer was assumed to be turbulent, starting at about 25% of the cone length, but it was considered laminar in all other calculations (see Refs. 1–3 of above paper).

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